On the Derivation of the Generalized Telegraphist's Equations for Full-Wave Analysis of Chirowaveguides

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Abstract—**This letter discusses a formulation of the generalized telegraphist's equations for the rigorous analysis of chirowaveguides that has recently been reported in the literature. It is found that when a chiral medium is in contact with the perfect electric conducting wall of the waveguide, the boundary conditions are not satisfied on the wall. A new formulation is proposed to overcome this limitation. Both formulations are applied to the calculation of the dispersion diagrams and field distributions of the completely filled parallel-plate chirowaveguide and the results are compared. Our proposed formulation is found to provide satisfactory results when analyzing such a structure.**

I. Introduction

In recent years considerable attention has been given to the study of wave propagation in chirowaveguides (CW's), i.e., waveguides containing chiral media. This growing interest stems from the potential application of these structures to the design of novel microwave and millimeter-wave components as well as from the possibility of developing new measurement techniques for the determination of material parameters of chiral composites [1]. Unfortunately, exact solutions are feasible only for a strictly limited number of structures, e.g., parallel-plate [2], [3] and circular CW's [4]. To overcome this limitation, a number of rigorous numerical techniques have been proposed [4]–[7].

This letter discusses a formulation of the generalized telegraphist's equations (GTE's) that has recently been proposed for the study of closed CW's [7]. It is shown that the GTE's derived in [7] are not valid for the full-wave analysis of CW's containing chiral media in contact with a perfect electric conducting (PEC) wall because the boundary conditions are not satisfied on such a wall. To overcome this problem we propose an alternative formulation of the GTE's. Both formulations are applied to the calculation of the dispersion diagrams and field distributions of the completely filled parallel-plate CW, and the results are compared with the exact solutions.

II. GTE's for Full-Wave Analysis of CW's

Consider a parallel-plate waveguide containing chiral media. We assume that the CW is uniform in the x direction and the electromagnetic field propagates along the z axis. For electromagnetic fields with a time dependence of the form $\exp(j\omega t)$, the constitutive relations in the chiral media are given by

$$\vec{D} = \varepsilon \vec{E} - \xi \vec{B} \tag{1a}$$

$$\vec{H} = \frac{\vec{B}}{\mu} - \xi \vec{E} \tag{1b}$$

where ε is the permittivity, μ the permeability, and ξ is the chirality admittance. In general, all three parameters are assumed to be complex functions of the y coordinate. For lossless chiral media, ε and μ are real parameters, while ξ is imaginary.

The procedure described in [7] to derive the GTE's of a CW can be briefly summarized in the following steps. First, the unknown $\vec{E}$ and $\vec{H}$ fields of the CW are expressed in terms of the normal modes of the corresponding empty waveguide, $\vec{e}$ and $\vec{h}$. Then, these modal expansions are substituted into Maxwell's equations and the Galerkin method is applied, resulting in a system of linear equations. Finally, the coefficients of the expansions of the longitudinal field components are eliminated, leading to a matrix eigenvalue equation for the propagation constants of the CW under analysis.

Since each basis function satisfies the boundary condition $\vec{h} \cdot \vec{a}_n = 0$ on the PEC wall of the empty waveguide, the boundary condition $\vec{H} \cdot \vec{a}_n = 0$ is enforced on the PEC wall of the CW. This boundary condition for the $\vec{H}$ field is correct provided that the medium in contact with the PEC wall is isotropic (achiral). However, for the chiral case, the boundary condition $\vec{B} \cdot \vec{a}_n = 0$ must be fulfilled on the PEC wall (it is clear from (1) that $\vec{B} \cdot \vec{a}_n = 0$ does not imply that $\vec{H} \cdot \vec{a}_n = 0$). Therefore, the GTE's derived in [7] are only valid when the medium in contact with the PEC wall of the CW is isotropic.

To overcome this limitation we propose the expansion of the $\vec{B}$ field instead of the $\vec{H}$ field. To illustrate this alternative formulation, we will consider the derivation of the GTE's for the parallel-plate CW. For this case, we can express the $\vec{E}$ and $\vec{B}$ fields in terms of the TEM, $\mathrm{TM}_{(n)}$ and $\mathrm{TE}_{[m]}$ modes of the empty parallel-plate waveguide as

$$\vec{E} = \sum_{j=1}^{\infty} V_{[j]} T'_{[j]} \vec{a}_x + \left[\frac{V_{(0)}}{\sqrt{b}} + \sum_{i=1}^{\infty} V_{(i)} T'_{(i)} \right] \vec{a}_y + \sum_{i=1}^{\infty} k_{c(i)} V^{z}_{(i)} T_{(i)} \vec{a}_z \tag{2a}$$

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$$\vec{B} = -\mu_0\left[\frac{I_{(0)}}{\sqrt{b}} + \sum_{i=1}^{\infty} I_{(i)}T'_{(i)}\right]\vec{a}_x + \mu_0\sum_{j=1}^{\infty} I_{[j]}T'_{[j]}\vec{a}_y + \mu_0\sum_{j=1}^{\infty} k_{c[j]}I^z_{[j]}T_{[j]}\vec{a}_z \tag{2b}$$

where $V_{[j]}$, $V_{(i)}$, $I_{[j]}$, $I_{(i)}$, $V^z_{(j)}$, and $I^z_{[j]}$ are the coefficients of the expansions, b is the distance between the PEC parallel plates, $k_{c(n)} = k_{c[n]} = n\pi/b$, and a spatial variation of the form $\exp(-\mathbf{j}\gamma z)$ is assumed in the z direction, where γ ($\gamma = \alpha + \mathbf{j}\beta$) denotes the propagation constant. The functions $T_{(n)}$ and $T_{[m]}$ are the generating functions of the $\mathrm{TM}_{(n)}$ and $\mathrm{TE}_{[m]}$ modes, respectively. They are given by

$$\begin{Bmatrix} T_{(n)} \\ T_{[m]} \end{Bmatrix} = \sqrt{\frac{2}{b}} \begin{Bmatrix} k_{c(n)}^{-1}\,\sin(k_{c(n)}y) \\ k_{c[m]}^{-1}\,\cos(k_{c[m]}y)\end{Bmatrix}. \tag{3}$$

The primes in (2) are used to denote derivative with respect to y.

For the sake of simplicity, in the following we assume non-permeable chiral media, i.e., $\mu = \mu_0$. Applying the procedure described previously, we obtain the following GTE's:

$$\gamma V_{(0)} = \mathbf{j}\omega\mu_0 I_{(0)} \tag{4a}$$

$$\gamma V_{(n)} = \frac{-\mathbf{j}k^2_{c(n)}}{\omega} \times \left[\sum_{j=1}^{\infty} V_{[j]}\left(2k^2_{c[j]}\int_0^b \frac{\xi}{\varepsilon}T_{[j]}T_{(n)}dy - \int_0^b \frac{\xi'}{\varepsilon}T'_{[j]}T_{(n)}dy\right) + \sum_{i=1}^{\infty} I_{(i)}k^2_{c(i)}\int_0^b \frac{T_{(i)}T_{(n)}}{\varepsilon}dy\right] + \mathbf{j}\omega\mu_0 I_{(n)} \tag{4b}$$

$$\gamma V_{[m]} = \mathbf{j}\omega\mu_0 I_{[m]} \tag{4c}$$

$$\gamma I_{(0)} = \frac{\mathbf{j}\omega}{\sqrt{b}}\left[\frac{V_{(0)}}{\sqrt{b}}\int_0^b \varepsilon dy + \sum_{i=1}^{\infty} V_{(i)}\int_0^b \varepsilon T'_{(i)}dy - 2\mu_0\sum_{j=1}^{\infty} I_{[j]}\int_0^b \xi T'_{[j]}dy\right] \tag{4d}$$

$$\gamma I_{(n)} = \mathbf{j}\omega\left[\frac{V_{(0)}}{\sqrt{b}}\int_0^b \varepsilon T'_{(n)}dy + \sum_{i=1}^{\infty} V_{(i)}\int_0^b \varepsilon T'_{(i)}T'_{(n)}dy - 2\mu_0\sum_{j=1}^{\infty} I_{[j]}\int_0^b \xi T'_{[j]}T'_{(n)}dy\right] \tag{4e}$$

$$\gamma I_{[m]} = \mathbf{j}\omega\left[2\mu_0\left(\frac{I_{(0)}}{\sqrt{b}}\int_0^b \xi T'_{[m]}dy + \sum_{i=1}^{\infty} I_{(i)}\int_0^b \xi T'_{(i)}T'_{[m]}dy\right) + \sum_{j=1}^{\infty} V_{[j]}\int_0^b \varepsilon T'_{[j]}T'_{[m]}dy\right]$$

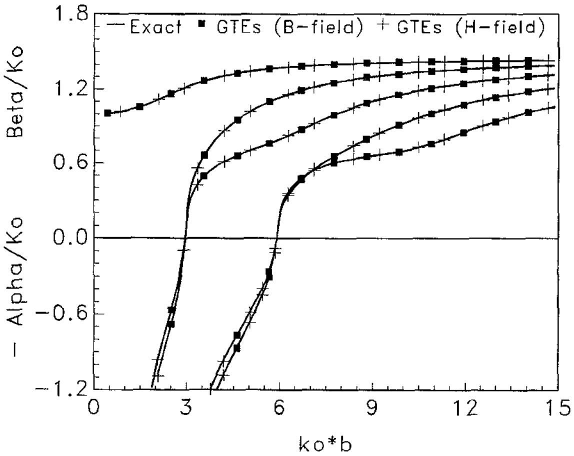

Fig. 1. Normalized propagation constant as a function of the normalized frequency for the completely filled parallel-plate CW with $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, and $\xi = \mathbf{j}\ 1$ mS.

$$+ \frac{\mathbf{j}}{\omega}\left[\sum_{j=1}^{\infty} V_{[j]}\left(2k^2_{c[j]}\int_0^b \frac{\xi\xi'}{\varepsilon}T_{[j]}T'_{[m]}dy - \int_0^b \frac{\xi'}{\varepsilon}T'_{[j]}T'_{[m]}dy\right) + \sum_{i=1}^{\infty} I_{(i)}k^2_{c(i)}\int_0^b \frac{\xi'}{\varepsilon}T_{(i)}T'_{[m]}dy - \frac{k^2_{c[m]}}{\mu_0}V_{[m]}\right] \tag{4f}$$

where the indexes n and m range from one to infinity. For practical purposes this doubly infinite system of linear equations is truncated to a finite number of basis functions, i.e., N $\mathrm{TM}_{(n)}$ modes and M $\mathrm{TE}_{[m]}$ modes. The eigenvalues of (4) are the propagation constants of the CW under analysis and the field distributions are obtained from the corresponding eigenvectors by using (2).

III. Numerical Results

To investigate the validity of the two formulations of the GTE's discussed above from a numerical viewpoint, we will consider the characterization of the completely filled parallel-plate CW. Exact results are available for this structure [2], [3].

Fig. 1 shows the normalized propagation constants of the first five modes as a function of the normalized frequency, $k_0 b$, where k_0 is the wavenumber of free-space. The value of the chirality admittance is $\xi = \mathbf{j}\ 1$ mS. The results obtained with both formulations are in very good agreement with the exact solutions.

However, the higher the value of ξ the worse the results obtained with the H-field formulation become. This is illustrated in Fig. 2, which shows the convergence curves calculated with the H-field formulation for $\xi = \mathbf{j}\ 5$ mS. It can be seen that the convergence is dramatically slow, and it becomes slower as the order of the mode increases. On the other hand, if the B-field

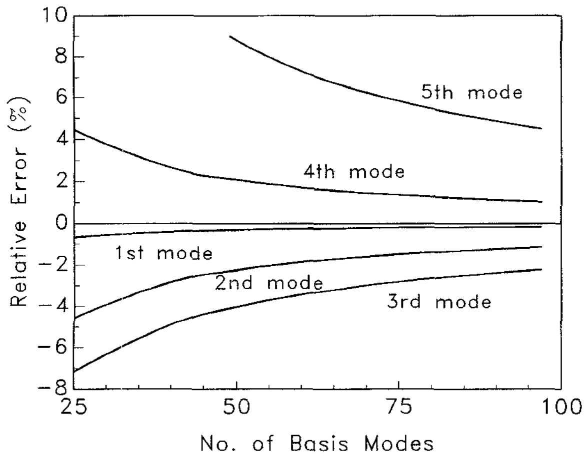

Fig. 2. Convergence curves calculated with the H-field formulation for the first five modes of the completely filled parallel-plate CW with $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, $\xi = \mathbf{j}$ 5 mS, and $k_0 b = 2.094$.

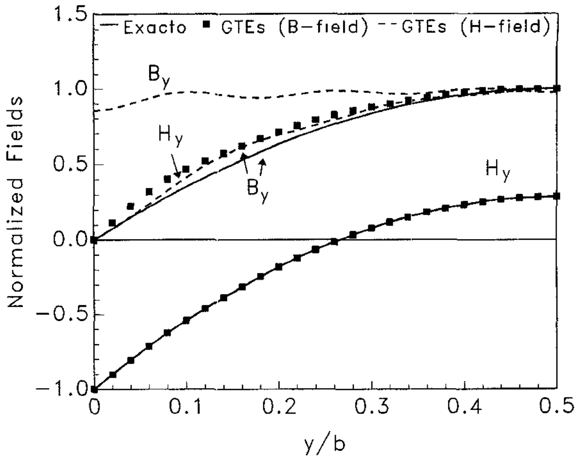

Fig. 3. Field distribution for the H_y and B_y components of the dominant mode of the completely filled parallel-plate CW with $\varepsilon = \varepsilon_0$, $\mu = \mu_0$, $\xi = \mathbf{j}$ 1 mS, and $k_0 b = 2.094$.

formulation is used, 21 basis modes are enough to obtain a relative error of less than 0.3% for the five modes considered.

Finally, Fig. 3 shows the field patterns of the H- and B-field components normal to the PEC plates of the CW, i.e., H_y and B_y. This figure shows the dominant mode, which is an even mode. Therefore, the field patterns are plotted between $y = 0$ and $y = b/2$. Each curve has been normalized to the maximum of its modulus. It can be seen that the exact value of H_y at the PEC plates ($y = 0$ and $y = b$) is nonzero; however, when the H-field formulation is used, this value is forced to zero. Thus, even for cases where the H-field formulation gives good results for the propagation constant the corresponding field patterns are wrong.

IV. Conclusion

In the previous derivation of the GTE's for full-wave analysis of CW's, the H field was taken as unknown and was therefore expressed in terms of the modes of the empty waveguide, so the boundary condition $\vec{H} \cdot \vec{a}_n = 0$ was enforced on the PEC wall of the CW. As a consequence, the resulting formulation is only valid when the medium in contact with the PEC wall of the CW is isotropic (achiral). However, if the B field is expanded instead of the H field, the more general boundary condition $\vec{B} \cdot \vec{a}_n = 0$ is fulfilled on the PEC wall of the CW. The validity of this latter formulation for the analysis of CW's has been demonstrated by applying it to the calculation of the dispersion diagrams and field distributions of the completely filled parallel-plate CW. By contrast, the formulation based on the H field leads to wrong results when analyzing this structure.

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